18 The Exterior Angle Theorem and Its Consequences

Definition. **(less than & greater than for line segments)** In a metric geometry, the line segment $\overline{AB}$ is less than (or smaller than) the line segment $\overline{CD}$ (written $\overline{AB} < \overline{CD}$) if $AB < CD$. $\overline{AB}$ is greater than (or larger than) $\overline{CD}$ if $AB > CD$. The symbol $\overline{AB} \leq \overline{CD}$ means that either $\overline{AB} < \overline{CD}$ or $\overline{AB} \cong \overline{CD}$.

Definition. **(less than & greater than for angles)** In a protractor geometry, the angle $\measuredangle ABC$ is less than (or smaller than) the angle $\measuredangle DEF$ (written $\measuredangle ABC < \measuredangle DEF$) if $m(\measuredangle ABC) < m(\measuredangle DEF)$. $\measuredangle ABC$ is greater than (or larger than) $\measuredangle DEF$ if $\measuredangle DEF < \measuredangle ABC$). The symbol $\measuredangle ABC \leq \measuredangle DEF$) means that either $\measuredangle ABC < \measuredangle DEF$) or $\measuredangle ABC \cong \measuredangle DEF$).

Theorem. In a metric geometry, $\overline{AB} < \overline{CD}$ if and only if there is a point $G \in \operatorname{int}(\overline{CD})$ with $\overline{AB} \cong \overline{CG}$.

1. Prove the above Theorem.

Theorem. In a protractor geometry, $\measuredangle ABC < \measuredangle DEF$) if and only if there is a point $G \in \operatorname{int}(\measuredangle DEF)$ with $\measuredangle ABC \cong \measuredangle DEG$).

2. Prove the above Theorem.

Definition. **(exterior angle, remote interior angle)** Given $\triangle ABC$ in a protractor geometry, if $A{-}C{-}D$ then $\measuredangle BCD$ is an exterior angle of $\triangle ABC$. $\measuredangle A$ and $\measuredangle B$ are the remote interior angles of the exterior angle $\measuredangle BCD$.

Theorem **(Exterior Angle Theorem)**. In a neutral geometry, any exterior angle of $\triangle ABC$ is greater than either of its remote interior angles.

3. Prove the above Theorem. [Th 6.3.3, p. 136]

4. In a protractor geometry prove the two exterior angles of $\triangle ABC$ at the vertex C are congruent.

5. In a neutral geometry prove that the base angles of an isosceles triangle are acute.

6. Show that at most one angle in triangle can be right or obtuse angle, and that at least two angles are acute.

Corollary In a neutral geometry, there is exactly one line through a given point P perpendicular to a given line ℓ.

7. Prove the above Corollary. [Cor 6.3.4, p. 137]

Theorem **(Side-Angle-Angle, SAA)**. In a neutral geometry, given two triangles $\triangle ABC$ and $\triangle DEF$, if $\overline{AB} \cong \overline{DE}$, $\measuredangle A \cong \measuredangle D$, and $\measuredangle C \cong \measuredangle F$, then $\triangle ABC \cong \triangle DEF$.

8. Prove the above Theorem. [Th 6.3.5, p 138]

We should note that the above proof (which is valid in any neutral geometry) is probably different from any you have seen before. In particular we did not prove $\measuredangle B \cong \measuredangle E$ by looking at the sums of the measures of the angles of the two triangles. We could not do this because we do not know any theorems about the sum of the measures of the angles of a triangle. **In particular the sum may not be the same for two triangles in an arbitrary neutral geometry**.

Theorem In a neutral geometry, if two sides of a triangle are not congruent, neither are the opposite angles. Furthermore, the larger angle is opposite the longer side.

9. Prove the above Theorem. [Th 6.3.6, p 138]

Theorem In a neutral geometry, if two angles of a triangle are not congruent, neither are the opposite sides. Furthermore, the longer side is opposite the larger angle.

10. Prove the above Theorem.

Theorem **(Triangle Inequality)**. In a neutral geometry the length of one side of a triangle is strictly less than the sum of the lengths of the other two sides.

11. Prove the above Theorem. [Th 6.3.8, p 139]

12. In a neutral geometry, if $D \in \operatorname{int}(\triangle ABC)$ prove that $AD + DC < AB + BC$ and $\measuredangle ADC > \measuredangle ABC$.

Theorem **(Open Mouth Theorem)**. In a neutral geometry, given two triangles $\triangle ABC$ and $\triangle DEF$ with $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$, if $\measuredangle B > \measuredangle E$ then $\overline{AC} > \overline{DF}$.

13. Prove the above Theorem. [Th 6.3.9, p 140]

Theorem In a neutral geometry, a line segment joining a vertex of a triangle to a point on the opposite side is shorter than the longer of the remaining two sides. More precisely, given $\triangle ABC$ with $\overline{AB} \leq \overline{CB}$, if $A - D - C$ then $\overline{DB} < \overline{CB}$.

14. Prove the above Theorem.

15. Prove the converse of Open Mouth Theorem: In a neutral geometry, given $\triangle ABC$ and $\triangle DEF$, if $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$ and $\overline{AC} > \overline{DF}$, then $\measuredangle B > \measuredangle E$.

16. In a neutral geometry, given $\triangle ABC$ such that the internal bisectors of $\measuredangle A$ and $\measuredangle C$ are congruent, prove that $\triangle ABC$ is isosceles.

17. Replace the word "neutral" in the hypothesis of Theorem 6.3.6 (Problem 9) with the word "protractor". Is the conclusion still valid?

19 Right Triangles

A word of caution is needed here. The first thing that many of us think about when we hear the phrase "right triangle" is the classical **Pythagorean Theorem**. This theorem is very much a Euclidean theorem. That is, it **is true in the Euclidean Plane but not in all neutral geometries** (see Problem 10). Thus in each proof of this section which deals with a general neutral geometry we must avoid the use of the Pythagorean Theorem.

Definition. **(right triangle, hypotenuse)** If an angle of $\triangle ABC$ is a right angle, then $\triangle ABC$ is a right triangle. A side opposite a right angle in a right triangle is called a hypotenuse.

Definition. **(the longest side, a longest side)** $\overline{AB}$ is the longest side of $\triangle ABC$ if $\overline{AB} > \overline{AC}$ and $\overline{AB} > \overline{BC}$. $\overline{AB}$ is a longest side of $\triangle ABC$ if $\overline{AB} \geq \overline{AC}$ and $\overline{AB} \geq \overline{BC}$.

Theorem In a neutral geometry, there is only one right angle and one hypotenuse for each right triangle. The remaining angles are acute, and the hypotenuse is the longest side of the triangle.

1. Prove the above Theorem. [Th 6.4.1, p 143]

Definition. **(legs)** If $\triangle ABC$ is a right triangle with right angle at C then the legs of $\triangle ABC$ are $\overline{AC}$ and $\overline{BC}$.

Theorem (Perpendicular Distance Theorem). In a neutral geometry, if ℓ is a line, $Q \in \ell$, and $P \notin \ell$ then (i) if $\overleftrightarrow{PQ} \perp \ell$ then $PQ \leq PR$ for all $R \in \ell$ (ii) if $PQ \leq PR$ for all $R \in \ell$ then $\overleftrightarrow{PQ} \perp \ell$.

2. Prove the above Theorem. [Th 6.4.2, p 144]

Definition. **(distance from P to ℓ)** Let ℓ be a line and P a point in a neutral geometry. If $P \notin \ell$, let Q be the unique point of ℓ such that $\overleftrightarrow{PQ} \perp \ell$. The distance from P to ℓ is

$$d(P,\ell) = \begin{cases} d(P,Q), & \text{if } P \notin \ell \\ 0, & \text{if } P \in \ell. \end{cases}$$

Theorem For any line ℓ in a neutral geometry and $P \notin \ell$ $\quad d(P,\ell) \leq d(P,R)$ for all $R \in \ell$. Furthermore, $d(P,\ell) = d(P,R)$ if and only if $\overleftrightarrow{PR} \perp \ell$.

Definition. **(altitude, foot of the altitude)** If ℓ is the unique perpendicular to $\overleftrightarrow{AB}$ through the vertex C of $\triangle ABC$ and if $\ell \cap \overleftrightarrow{AB} = \{D\}$, then $\overline{CD}$ is the altitude from C. D is the foot of the altitude (or of the perpendicular) from C.

Theorem In a neutral geometry, if $\overline{AB}$ is a longest side of $\triangle ABC$ and if D is the foot of the altitude from C, then $A - D - B$.

3. Prove the above Theorem. [Th 6.4.3, p 145]

Theorem (Hypotenuse-Leg, HL). In a neutral geometry if $\triangle ABC$ and $\triangle DEF$ are right triangles with right angles at C and F, and if $\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{DF}$, then $\triangle ABC \cong \triangle DEF$.

4. Prove the above Theorem. [Th 6.4.4, p 146]

Theorem (Hypotenuse-Angle, HA). In a neutral geometry, let $\triangle ABC$ and $\triangle DEF$ be right triangles with right angles at C and F. If $\overline{AB} \cong \overline{DE}$ and $\measuredangle A \cong \measuredangle D$, then $\triangle ABC \cong \triangle DEF$.

Definition. **(perpendicular bisector)** The perpendicular bisector of the segment $\overline{AB}$ in a neutral geometry is the (unique) line ℓ through the midpoint M of $\overline{AB}$ and which is perpendicular to $\overline{AB}$.

Theorem In a neutral geometry the perpendicular bisector ℓ of the segment $\overline{AB}$ is the set $\mathcal{B} = \{P \in \mathcal{S} \mid AP = BP\}$.

5. Prove the above Theorem. [Th 6.4.6, p 147]

6. In a neutral geometry, if D is the foot of the altitude of $\triangle ABC$ from C and $A - B - D$, then prove $\overline{CA} > \overline{CB}$.

7. In a neutral geometry, denote by M_1 the foot of the altitude of $\triangle ABM$ from M and let $A - M_1 - B$. Prove that then $\overline{MA} > \overline{MB}$ if and only if $\overline{M_1A} > \overline{M_1B}$.

8. If M is the midpoint of $\overline{BC}$ then $\overline{AM}$ is called a **median** of $\triangle ABC$. Consider $\triangle ABC$ such that $\overline{AB} < \overline{AC}$. Let E, D and H denote the points in which bisector of angle, median and altitude from A intersect line $\overleftrightarrow{BC}$, respectively. Show that (a) $\measuredangle AEB < \measuredangle AEC$; (b) $\overline{BE} < \overline{CE}$; (c) we have $H - E - D$.

9. (a.) Prove that in a neutral geometry if $\triangle ABC$ is isosceles with base $\overline{BC}$ then the following are collinear: (i) the median from A; (ii) the bisector of $\measuredangle A$; (iii) the altitude from A; (iv) the perpendicular bisector of $\overline{BC}$. (b.) Conversely, in a neutral geometry prove that if any two of (i)-(iv) are collinear then the triangle is isosceles (six different cases).

10. Show that the conclusion of the Pythagorean Theorem is not valid in the Poincaré Plane by considering $\triangle ABC$ with $A(2,1)$, $B(0,\sqrt{5})$, and $C(0,1)$. Thus the Pythagorean Theorem does not hold in every neutral geometry.

Theorem In a neutral geometry, if $\overrightarrow{BD}$ is the bisector of $\measuredangle ABC$ and if E and F are the feet of the perpendiculars from D to $\overleftrightarrow{BA}$ and $\overleftrightarrow{BC}$ then $\overline{DE} \cong \overline{DF}$.

11. Prove the above Theorem. [Th 6.4.7, p 148]

20 Circles and Their Tangent Lines

Definition. **(circle with center C and radius r, chord, diameter, radius segment).** If C is a point in a metric geometry $(\mathcal{S}, \mathcal{L}, d)$ and if $r > 0$, then

$$\mathcal{C} = \mathcal{C}_r(C) = \{P \in \mathcal{S} \mid PC = r\}$$

is a circle with center C and radius r. If A and B are distinct points of $\mathcal{C}$ then $\overline{AB}$ is a chord of $\mathcal{C}$. If the center C is a point on the chord $\overline{AB}$, then $\overline{AB}$ is a diameter of $\mathcal{C}$. For any $Q \in \mathcal{C}$, $\overline{CQ}$ is called a radius segment of $\mathcal{C}$.

1. Find and sketch the circle of radius 1 with center $(0,0)$ in the Euclidean Plane and in the Taxicab Plane. [Ex 6.5.1, p150]

2. Consider $\{\mathbb{R}^2, \mathcal{L}_E\}$ with the max distance d_s (recall $d_s(P,Q) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$ where $P(x_1, y_1)$ and $Q(x_2, y_2)$ denote two points in $\mathbb{R}^2$). Sketch the circle $\mathcal{C}_1((0,0))$.

3. Show that $\mathcal{A} = \{(x,y) \in \mathbb{H} \mid x^2 + (y-5)^2 = 16\}$ is the Poincaré circle $\mathcal{C}$ with center $(0,3)$ and radius $\ln 3$. [Ex 6.5.2, p151]

Our first result tells us that in a neutral geometry the center and radius of a circle are determined by any three points on the circle.

Theorem. In a neutral geometry, let $\mathcal{C}_1 = \mathcal{C}_r(C)$ and $\mathcal{C}_2 = \mathcal{C}_s(D)$. If $\mathcal{C}_1 \cap \mathcal{C}_2$ contains at least three points, then $C = D$ and $r = s$. Thus, three points of a circle in a neutral geometry uniquely determine that circle.

4. Prove the above Theorem. [Th 6.5.3, p152]

Corollary. For any circle in a neutral geometry, the perpendicular bisector of any chord contains the center.

5. If $\overline{AB}$ is a chord of a circle in a neutral geometry but is not a diameter, prove that the line through the midpoint of $\overline{AB}$ and the center of the circle is perpendicular to $\overline{AB}$.

6. Prove that a line in a neutral geometry intersects a circle at most twice.

Definition. **(interior, exterior).** Let $\mathcal{C}$ be the circle with center C and radius r. The interior of $\mathcal{C}$ is the set $\text{int}(\mathcal{C}) = \{P \in \mathcal{S} \mid CP < r\}$. The exterior of $\mathcal{C}$ is the set $\text{ext}(\mathcal{C}) = \{P \in \mathcal{S} \mid CP > r\}$.

Theorem. If $\mathcal{C}$ is a circle in a neutral geometry then $\text{int}(\mathcal{C})$ is convex.

7. Prove the above Theorem. [Th 6.5.5, p153]

Definition. **(tangent, point of tangency).** In a metric geometry, a line ℓ is a tangent to the circle $\mathcal{C}$ if $\ell \cap \mathcal{C}$ contains exactly one point (which is called the point of tangency). ℓ is called a secant of the circle $\mathcal{C}$ if $\ell \cap \mathcal{C}$ has exactly two points.

8. In the Taxicab Plane prove that for the circle $\mathcal{C} = \mathcal{C}_1((0,0))$: (a). There are exactly four points at which a tangent to $\mathcal{C}$ exists. (b). At each point in part (a) there are infinitely many tangent lines.